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Anisotropic tensor spaces and functions, the mean values of tensor quantities and the limiting criteria☆

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Abstract

The concept of an anisotropic vector space with a tensor basis which is invariant under a symmetry transformations of a three-dimensional Euclidean vector space is introduced using the example of symmetric second- and fourth-rank Euclidean tensors. In addition to the traditional operation of summation, the operation of multiplication in a fixed tensor basis is introduced for the elements of this space, that is, the axioms of a ring with an identity element and zero divisors, which enable one to carry out algebraic and functional operations. The possibilities of the proposed mathematical procedure are illustrated using examples of anisotropic tensor functions of a tensor argument, by the general solution of the classical problem of calculating the mean value of the tensor of the moduli of elasticity of a single-phase grain-oriented polycrystalline material and the construction of the strength surfaces of anisotropic composite materials.

1. Anisotropic tensor spaces

Following the well-known approach,[1] we consider a six-dimensional space of symmetric second-rank tensors

$$S \equiv \mathrm{sym} E \otimes E \subset T_2 \equiv E \otimes E$$

Here E are the vectors of a three-dimensional Euclidean vector space, E^3. In the six-dimensional space S, and an orthonormalized basis $\omega^{(k)}$ ($k = 1, 2, \ldots, 6$) exists such that any symmetric second-rank tensor α can be represented in the form

$$\alpha = \sum \alpha_k \omega^{(k)}, \quad \omega^{(k)} \cdot \omega^{(l)} = \omega_{ij}^{(k)} \omega_{ij}^{(l)} = \delta_{kl}$$

where $\alpha_k \in R$ are the coordinates of the symmetric tensor in the given tensor basis. Here,

$$\alpha_k = \alpha \cdot \omega^{(k)}$$

Here and everywhere henceforth, k takes the values of 1, 2, . . ., 6 and a summation sign, unless otherwise stated, denotes summation from $k = 1$ to $k = 6$,

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In addition to the linear operations of the summation of tensors and the multiplication of a tensor by a number, the operation of the multiplication of two tensors in a fixed basis $\omega^{(k)}$ is introduced.

Definition 1. The tensor $\alpha\beta = \sum \alpha_k \beta_k \omega^{(k)} \in S$, where α_k and β_k are the coordinates of the tensors α and β in the basis $\omega^{(k)}$, is called the product of two symmetric tensors of the second rank α and β in the basis $\omega^{(k)}$.

Definition 2. The tensor $\omega = \sum \omega^{(k)}$ is called the director of the orthonormal basis $\omega^{(k)}$.

The basis $\omega^{(k)}$ is invariant under the symmetry transformations of the vector space E^3.

1.1. The case of orthotropic symmetry

$$\omega^{(m)} = \mathrm{diag}(\omega_1^{(m)}, \omega_2^{(m)}, \omega_3^{(m)}), \quad m = 1, 2, 3$$

Using the quaternion representation of the numerical triplets $(\omega_1^{(k)}, \omega_2^{(k)}, \omega_3^{(k)})$, which constitute an orthonormalized basis in the space R^3,[2] we have

$$\omega^{(1)} = \mathrm{diag}(p_0^2 + p_1^2 - p_2^2 - p_3^2, 2(p_1p_2 - p_0p_3), 2(p_0p_2 + p_1p_3))$$

$$\omega^{(2)} = \mathrm{diag}(2(p_0p_3 + p_1p_2), p_0^2 - p_1^2 + p_2^2 - p_3^2, 2(p_2p_3 - p_0p_1))$$

$$\omega^{(3)} = \mathrm{diag}(2(p_1p_3 - p_0p_2), 2(p_0p_1 + p_2p_3), p_0^2 - p_1^2 - p_2^2 + p_3^2)$$

Here, $p_0^2 + p_1^2 + p_2^2 + p_3^2 = 1$.

1.2. The case of orthotropy accompanying bulk isotropy (one of the basis tensors is spherical)

$$\omega^{(1)} = \frac{\mathrm{diag}(1, 1, 1)}{\sqrt{3}}, \quad \omega^{(2)} = \frac{\mathrm{diag}(1, q_2, -1 - q_2)}{\sqrt{2}\sqrt{1 + q_2 + q_2^2}}, \quad \omega^{(3)} = \frac{\mathrm{diag}(1, q_3, -1 - q_3)}{\sqrt{2}\sqrt{1 + q_3 + q_3^2}}$$

where $q_{2,3} = t \pm \sqrt{1 + t + t^2}$

1.3. The case of tetragonal symmetry and transverse isotropy

$$\omega^{(1)} = \frac{\mathrm{diag}(1, 1, q_1)}{\sqrt{2 + q_1^2}}, \quad \omega^{(2)} = \frac{\mathrm{diag}(1, 1, q_2)}{\sqrt{2 + q_2^2}}, \quad \omega^{(3)} = \frac{\mathrm{diag}(1, -1, 0)}{\sqrt{2}}$$

where $q_{1,2} = t \pm \sqrt{2 + t^2}$.

1.4. The case of cubic symmetry

$$\omega^{(1)} = \frac{\mathrm{diag}(1, 1, 1)}{\sqrt{3}}, \quad \omega^{(2)} = \frac{\mathrm{diag}(1, 1, -2)}{\sqrt{6}}, \quad \omega^{(3)} = \frac{\mathrm{diag}(1, -1, 0)}{\sqrt{2}}$$

In all of the above cases

$$\omega^{(4)} = \frac{1}{\sqrt{2}}\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}, \quad \omega^{(5)} = \frac{1}{\sqrt{2}}\begin{Vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad \omega^{(6)} = \frac{1}{\sqrt{2}}\begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

It can be shown by direct verification that the bases $\omega^{(k)}$ which have been obtained are orthonormalized.

It can also be shown that, in the case of the corresponding symmetry transformations of the space E^3, the tensor product does not change and the scalar product $\alpha - \beta$ is the scalar product of the tensor $\alpha\ \beta$ with the director of the tensor basis, that is,

$$\alpha \cdot \beta = (\alpha\beta) \cdot \omega$$

Definition 3. The tensor space S, in which the operation of the multiplication of two tensors in a fixed tensor basis was introduced, is called an anisotropic tensor space $\tilde{S}$.

The axioms of an associative-commutative ring with an identity element ω and zero divisors are satisfied in the space $\tilde{S}$.

Moreover, for elements which are not zero divisors,

$$\forall\alpha \in \tilde{S}(\alpha_1\alpha_2\ldots\alpha_6 \neq 0)\ \exists\alpha^{-1} : \alpha\alpha^{-1} = \omega;\ \frac{\alpha}{\beta} = \alpha\beta^{-1}\ (\beta_1\beta_2\ldots\beta_6 \neq 0)$$

The director of the basis is the tensor identity element of the space $\tilde{S}$ and, by virtue of the axioms which have been adopted in this space, it is possible to carry out algebraic and functional operations.

The algebraic identities

$$(\alpha+\beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2;\quad \alpha^{-1} - (\alpha+\omega)^{-1} = \alpha^{-1}(\alpha+\omega)^{-1},\quad \alpha_1\alpha_2\ldots\alpha_6 \neq 0,\ \alpha \neq -\omega$$

are easily verified.

In a similar manner, an anisotropic space $\tilde{T}_4 \equiv \tilde{S} \otimes \tilde{S}$ of fourth-rank tensors, which are symmetrical with respect to the first two and the last two indices and the extreme pairs of indices, can be defined. The system $\omega^{(k)} \otimes \omega^{(k)}$ and the tensor identity element, that is, the tensor

$$I = \sum \omega^{(k)} \otimes \omega^{(k)},\quad \text{or}\quad I_{ijmn} = \sum \omega_{ij}^{(k)}\omega_{mn}^{(k)}$$

is the basis of this space, which is invariant under symmetry transformations of the vector space E^3. In this case,

$$\omega^{(k)} \otimes \omega^{(k)} \cdot \omega^{(l)} \otimes \omega^{(l)} = \omega_{ij}^{(k)}\omega_{mn}^{(k)}\omega_{ij}^{(l)}\omega_{mn}^{(l)} = \delta_{kl}$$

2. Anisotropic tensor functions of a tensor argument

We will call the set of tensors

$$\alpha = \sum \alpha_k\omega^{(k)},\quad \alpha_k \in M_k \subset R$$

the domain $D \subset \tilde{S}$ of the anisotropic tensor space $\tilde{S}$.

We shall say that an anisotropic tensor function $f(\alpha)$ of corresponding symmetry is defined in the domain D of the space $\tilde{S}$ if a law is stated according to which a tensor $f(\alpha) \in \tilde{S}$ is set in correspondence to each tensor α from D. In the basis $\omega^{(k)}$, this law can be written in the form

$$f(\alpha) = \sum \varphi_k(\alpha_1, \alpha_2, \ldots, \alpha_6)\omega^{(k)},\quad \varphi_k(\alpha_1, \alpha_2, \ldots, \alpha_6) = f(\alpha)\cdot\omega^{(k)}$$

where $\varphi_k(\alpha_1, \alpha_2, \ldots, \alpha_6)$ are scalar functions which are defined when $\alpha_k \in M_k$.

We now introduce elementary anisotropic tensor functions of a tensor argument into the treatment as a generalization of conventional elementary functions. A power function is introduced by the equality

$$\alpha^p = \sum \alpha_k^p\omega^{(k)},\quad p \in R$$

Logarithmic, basic exponential and trigonometric function

$$\ln\alpha = \sum \ln\alpha_k\omega^{(k)},\quad e^{\alpha} = \sum e^{\alpha_k}\omega^{(k)},\quad \sin\alpha = \sum \sin\alpha_k\omega^{(k)},\ \cos\alpha = \sum \cos\alpha_k\omega^{(k)}$$

are introduced in a similar way

A rational function is introduced by the equality

$$\frac{P_n(\alpha)}{Q_m(\alpha)} = \sum \frac{P_n(\alpha_k)}{Q_m(\alpha_k)}\omega^{(k)}$$

where n and m are integral powers of tensor polynomials and $P_n(\alpha_k)$, $Q_m(\alpha_k)$ are polynomials above the field of real numbers. This function is not defined for values of α_k, which are roots of the equations $Q_m(\alpha_k)=0$.

It can be shown that properties which are similar to those of ordinary elementary functions are satisfied in the case of the elementary anisotropic tensor functions which have been introduced.

An anisotropic tensor function of a tensor argument in the space $\tilde{T}_4$ is defined in a similar way

$$f(\alpha) = \sum \varphi_k(\alpha_1, \alpha_2, \dots, \alpha_6)\omega^{(k)} \otimes \omega^{(k)}; \quad \varphi_k(\alpha_1, \alpha_2, \dots, \alpha_6) = f(\alpha) \cdot \omega^{(k)} \otimes \omega^{(k)},$$
$$\alpha \in \tilde{T}, \quad f(\alpha) \in \tilde{T}$$

3. Mean values of tensor quantities

We shall illustrate the possibilities of the proposed mathematical apparatus taking the example of the solution of the classical problem of calculating the mean value of the tensor of the moduli of elasticity of a single-phase grain-orientated polycrystalline material.[3,4] When the operations which have been introduced are taken into account, the corresponding mean can be determined by the equality

$$c^{(p)} = \left(\int_G (Q(g) * c)^p F(g) dg\right)^{1/p}, \quad p \in R; \quad \int_G F(g) dg = 1$$

Here $Q * c \leftrightarrow Q_{im}Q_{jn}Q_{pr}Q_{qs}c_{mnrs}$ is the orbit of the tensor c, $||Q_{ij}||$ is the matrix of the direction cosines (the rotation matrix) which defines the position of the crystallographic axes of the grains of the polycrystalline material, g is an element of the space of the rotations G and $F(g)$ is a grain-orientation function.

When $p=1$, we have the mean value of the tensor which corresponds to the Voigt model, when $p=-1$, to the Reuss model and, when $p\to 0$, the geometric mean, which is independent of whether the tensor of the moduli of elasticity or the tensor of the compliances, which is inverse to it, is averaged.

We now consider the determination of the geometric mean of the moduli of elasticity tensor in detail. As in the case of scalar quantities, this means that

$$\ln c^{(0)} = \int_G Q(g) * \ln c F(g) dg$$

When account is taken of the representation of logarithmic tensor functions, we have

$$\sum \ln\lambda_l^{(0)} \tilde{\omega}^{(l)} \otimes \tilde{\omega}^{(l)} = \int_G Q(g) * \sum (\ln\lambda_k \omega^{(k)} \otimes \omega^{(k)}) F(g) dg = \sum \ln\lambda_k \langle Q * \omega^{(k)} \otimes \omega^{(k)} \rangle$$

Averaging over a set of all orientations of the grains of the polycrystalline material is denoted by angular brackets, $\omega^{(k)} \otimes \omega^{(k)}$ and $\tilde{\omega}^{(l)} \otimes \tilde{\omega}^{(l)}$ are basis tensors corresponding to the symmetry of the grains and the polycrystalline material and λ_k and $\lambda_l^{(0)}$ are the Kelvin–Rychlewski[1] moduli of a grain and of the material.

Taking account of the orthogonality of the basis tensors, we have

$$\ln\lambda_l^{(0)} = \sum \ln\lambda_k \langle Q * \omega^{(k)} \otimes \omega^{(k)} \rangle \cdot \tilde{\omega}^{(l)} \otimes \tilde{\omega}^{(l)}$$

or

$$\lambda_l^{(0)} = \lambda_1^{p_{1l}} \lambda_2^{p_{2l}} \dots \lambda_6^{p_{6l}}, \quad p_{kl} = \langle Q * \omega^{(k)} \otimes \omega^{(k)} \rangle \cdot \tilde{\omega}^{(l)} \otimes \tilde{\omega}^{(l)}$$

It can be seen from this relation that the geometric mean moduli of elasticity are determined by the moduli of elasticity of a grain and the weighting factors p_{kl}, which depend on the structure of the basis tensors. In the case of cubic symmetry of the tensor c and orthotropic symmetry of the tensor $c^{(0)}$, we have

$$\lambda_1^{(0)} = 3K$$

$$\lambda_{2,3}^{(0)} = \lambda_2^{1-\chi_{2,3}}\lambda_4^{\chi_{2,3}}, \quad \chi_{2,3} = 3\Delta_1 - \Delta_2 + \Delta_3 - 2q_{2,3}(\Delta_2 - \Delta_3)$$

$$\lambda_4^{(0)} = \lambda_2^{\chi_4}\lambda_4^{1-\chi_4}, \quad \chi_4 = 2\Delta_2 + 2\Delta_3 - 2\Delta_1$$

$$\lambda_5^{(0)} = \lambda_2^{\chi_5}\lambda_4^{1-\chi_5}, \quad \chi_5 = 2\Delta_3 + 2\Delta_1 - 2\Delta_2$$

$$\lambda_6^{(0)} = \lambda_2^{\chi_6}\lambda_4^{1-\chi_6}, \quad \chi_6 = 2\Delta_1 + 2\Delta_2 - 2\Delta_3$$

$$\Delta_i = \langle Q_{i1}^2Q_{i2}^2 + Q_{i2}^2Q_{i3}^2 + Q_{i3}^2Q_{i1}^2\rangle, \quad i = 1, 2, 3; \quad 0 \le \Delta_i \le 1/3$$

$$q_{2,3} = t \pm \sqrt{1 + t + t^2}, \quad t = \frac{\Delta_1 - \Delta_2}{\Delta_3 - \Delta_2}$$

Here $K = c_{11} + 2c_{12}$ is the bulk modulus, $\lambda_2 = c_{11} - c_{12}$, $\lambda_4 = 2c_{44}$, c_{ij} are the moduli of elasticity in matrix notation and Δ_i are grain-orientation parameters.

In the case of an equiprobable distribution of the crystallographic axes (a quasi-isotropic polycrystalline material)

$$\Delta_1 = \Delta_2 = \Delta_3 = 1/5$$

For a distribution of the crystallographic axes which is axisymmetric about the Ox_3 axis (a macroscopically transversely-isotropic polycrystalline material), the condition

$$\Delta_1 = \Delta_2 = (1 + 3\Delta_3)/8$$

is satisfied.

In this case,

$$\lambda_2^{(0)} = \lambda_2^{1-3\Delta_3}\lambda_4^{3\Delta_3}, \quad \lambda_4^{(0)} = \lambda_5^{(0)} = \lambda_2^{2\Delta_3}\lambda_4^{1-2\Delta_3}, \quad \lambda_3^{(0)} = \lambda_6^{(0)} = \lambda_2^{(1-\Delta_3)/2}\lambda_4^{(1+\Delta_3)/2}$$

For a quasi-isotropic material, when $\Delta_3 = 1/5$, the well-known result[5]

$$\lambda_2^{(0)} = \lambda_3^{(0)} = \lambda_4^{(0)} = \lambda_5^{(0)} = \lambda_6^{(0)} = \lambda_2^{2/5}\lambda_4^{3/5}, \quad \text{или} \quad c_{44}^{(0)} = c_{44}A^{-2/5}$$

$$A = 2c_{44}(c_{11} - c_{12})^{-1} = \lambda_4/\lambda_2$$

follows from these relations, where A is the elastic anisotropy parameter of a single crystal.

For the special case of the distribution of the crystallographic axes when $\Delta_1 = \Delta_2 = 1/4$, $\Delta_3 = 0$, the results obtained here is identical to the well-known exact solution within the framework of the model of a polycrystalline material being considered.[6]

The geometric mean moduli of elasticity tensors and the compliances of a macroscopically orthotropic polycrystalline material are found using the expansions

$$c^{(0)} = \sum \lambda_k^{(0)}\tilde{\omega}^{(k)} \otimes \tilde{\omega}^{(k)}, \quad s^{(0)} = \sum (\lambda_k^{(0)})^{-1}\tilde{\omega}^{(k)} \otimes \tilde{\omega}^{(k)}$$

where $\tilde{\omega}^{(k)}$ are the basis tensors of the space $\tilde{S}$ corresponding to orthotropy with bulk isotropy.

4. Limiting criteria

We will now use the proposed mathematical apparatus to obtain phenomenological limiting criteria of certain anisotropic materials. Considering the stress space $\sum$, the elements of which are stress tensors at a given point of an anisotropic body ($\sigma \in \Sigma \subset \tilde{S}$), we will represent the limiting surface by the equality

$$f(\sigma) \cdot \omega = 1$$

In the case of orthotropic materials, on representing, in particular, the function $f(\sigma)$ by a second-degree tensor polynomial

$$f(\sigma) = \alpha\sigma^2 + \beta\sigma; \quad \alpha = \sum \alpha_k \omega^{(k)}, \quad \beta = \sum \beta_k \omega^{(k)}, \quad \sigma = \sum \sigma_k \omega^{(k)}, \quad \sigma^2 = \sum \sigma_k^2 \omega^{(k)}$$

and superposing the basis vectors of the space E^3 with the principal axes of anisotropy, we have

$$\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \ldots + \alpha_6\sigma_6^2 + \beta_1\sigma_1 + \beta_2\sigma_2 + \ldots + \beta_6\sigma_6 = 1$$

Under the assumption that the limiting state is invariant under a change in the specified direction of shear to the opposite direction, we have

$$\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \ldots + \alpha_6\sigma_6^2 + \beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3 = 1$$

This phenomenological equation of the limiting surface contains nine dimensional material constants and three dimensionless parameters, which determine the form of the basis tensors $\omega^{(1)}$, $\omega^{(2)}$, $\omega^{(3)}$ of the anisotropic tensor space. With additional hypotheses of a physical nature, the number of parameters which are subject to experimental determination can be reduced.

In the case of a spatially-reinforced composite material of cubic symmetry, for which the direction of the reinforcement coincides with the three-and four-fold symmetry axes of a cube in E^3, when account is taken of possible fracture by different mechanisms (fracture of the reinforcing fibres under tension or the loss of their stability under compression), we arrive, in the simplest case, at a four-constant strength surface in six-dimensional stress space

$$\alpha_1\sigma_1^2 + \alpha_2\sigma_2^2 + \ldots + \alpha_6\sigma_6^2 + \beta_1\sigma_1 = 1, \quad \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_5 = \alpha_6$$

$$\sigma_1 = \sigma\cdot\omega^{(1)} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{\sqrt{3}}, \quad \sigma_2 = \sigma\cdot\omega^{(2)} = \frac{\sigma_{11} + \sigma_{22} - 2\sigma_{33}}{\sqrt{6}}$$

$$\sigma_3 = \sigma\cdot\omega^{(3)} = \frac{\sigma_{11} - \sigma_{22}}{\sqrt{2}}$$

$$\sigma_4 = \sigma\cdot\omega^{(4)} = \sqrt{2}\sigma_{23}, \quad \sigma_5 = \sigma\cdot\omega^{(5)} = \sqrt{2}\sigma_{31}, \quad \sigma_6 = \sigma\cdot\omega^{(6)} = \sqrt{2}\sigma_{12}$$

The physical meaning of the material constants of this equations becomes clear if one considers the four independent stress states

1) $\sigma_{11} = \sigma_{22} = \sigma_{33} = p_+, \quad \sigma_{23} = \sigma_{31} = \sigma_{12} = 0;$

2) $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p_-, \quad \sigma_{23} = \sigma_{31} = \sigma_{12} = 0;$

3) $\sigma_{11} = -\sigma_{22} = \tau_1, \quad \sigma_{33} = \sigma_{23} = \sigma_{31} = \sigma_{12} = 0;$

4) $\sigma_{23} = \tau_2, \quad \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{31} = \sigma_{12} = 0.$

Here, p_+ and p_- are the limit stresses under isotropic tension and compression, and τ_1 and τ_2 are the limit stresses under simple shear in a plane which passes through the two- and four-fold axes of symmetry in the directions of the two- and four-fold axes respectively.

Substituting these relations into the strength surface equation we obtain

$$3\alpha_1 p_+^2 + \sqrt{3}\beta_1 p_+ = 1, \quad 3\alpha_1 p_-^2 - \sqrt{3}\beta_1 p_- = 1, \quad 2\alpha_2\tau_1^2 = 1, \quad 2\alpha_4\tau_2^2 = 1$$

whence

$$\alpha_1 = \frac{1}{3p_+p_-}, \quad \beta_1 = \frac{p_- - p_+}{\sqrt{3}p_+p_-}, \quad \alpha_2 = \frac{1}{2\tau_1^2}, \quad \alpha_4 = \frac{1}{2\tau_2^2}$$

When the limiting state is independent of the spherical part of the stress tensor in the case of the condition $\tau_1 = \tau_2 = \tau$, on changing to a five-dimensional pure shear space,[7] we obtain the simplest strength surface of an isotropic material in the form of the equation of a sphere in five-dimensional space

$$\sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 + \sigma_6^2 = 2\tau^2$$

which corresponds to the widely used energy theory of strength or the Huber-Mises-Hencky yield condition in the mathematical theory of plasticity.

References

1. Rychlewski J. Hooke's law. *Prikl Mat Mekh* 1984;**48**(3):420–5.
2. Borisov AV, Mamayev IS. Dynamics of solids. In: *Regulyar naya khaoticheskaya dynamica.* Izhevsk: NITs; 2001.
3. Shermergor TD. *The Theory of Elasticity of Micro-inhomogeneous Media*. Moscow: Nauka; 1977.
4. Matthis S, Humbert M. The realization of the concept of a geometric mean for calculation physical constants of polycrystalline materials. *Phys Stat Solidi (b)* 1993;**177**:K47–50.
5. Aleksandrov KS. Mean values of tensor quantities. *Dokl Akad Nauk SSSR* 1966;**164**(4):800–4.
6. Berestova SA, Mityushov YeA. Exact solution of the problem of determining the effective moduli of elasticity of micro-inhomogeneous media. *Prikl Mat Mekh* 1999;**63**(3):524–7.
7. Il'yushin AA. *Plasticity*. Moscow: Izd Akad Nauk SSSR; 1963.

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